FST 3-7 Notes
Topic: Composition of Functions
GOAL:
Formalize the concept of composition of functions by defining composition and introducing the $\circ$ symbol.

SPUR Objectives

A Find equations for and values of composites of functions.

F Identify properties of composites of functions.

Vocabulary

composite
function composition

Definition of Composite Function

Suppose f and g are functions. The **composite** of g with f, written $g \circ f$, is the function defined by

$$(g \circ f)(x) = g(f(x)).$$

The domain of $g \circ f$ is the set of values of x in the domain of f for which $f(x)$ is in the domain of g.

* Composition of Functions is not commutative (order matters!)

Example 1: Let $f(x) = x^2$ *and* $g(x) = \dfrac{1}{3x+1}$. Evaluate.

a) $f(g(4))$

$f\left(\frac{1}{3(4)+1}\right)$

$f\left(\frac{1}{13}\right)$

$= \left(\frac{1}{13}\right)^2 = \frac{1^2}{13^2}$

$f(g(4)) = \frac{1}{169}$

b) $g(f(4))$

$g(4^2)$

$g(16)$

$= \frac{1}{3(16)+1}$

$g(f(4)) = \frac{1}{49}$

c) $(f \circ g)(4) = f(g(4))$

$f\left(\frac{1}{3(4)+1}\right)$

$f\left(\frac{1}{13}\right)$

$= \left(\frac{1}{13}\right)^2$

$= \frac{1}{169}$

On your own:

Let $f(x) = 3x^2 - 3x$ *and* $g(x) = x + 7$. Evaluate:

a) $(f \circ g)(3)$ $= f(g(3))$

$f(3+7) = f(10)$

$f(10) = 3(10)^2 - 3(10)$

$= 300 - 30$

$f(g(3)) = 270$

b) $g(f(3))$

$g(3(3)^2 - 3(3))$

$g(27-9) = g(18)$

$g(18) = 18+7$

$g(f(3)) = 25$

Example 2: Let $f(x) = x^2$ *and* $g(x) = \dfrac{1}{3x+1}$.

a) Derive a formula for $(f \circ g)(x)$ $= f(g(x))$

$f\left(\frac{1}{3x+1}\right) \rightarrow \left(\frac{1}{3x+1}\right)^2 = \frac{1^2}{(3x+1)^2} = \frac{1}{(3x+1)(3x+1)}$

FOIL

$= \frac{1}{9x^2+3x+3x+1} = \frac{1}{9x^2+6x+1}$

b) Give a simplified formula for $(g \circ f)(x)$ $g(f(x))$

$g(x^2) \rightarrow \frac{1}{3x^2+1}$

c) Verify that $f \circ g \neq g \circ f$ by graphing.

$\frac{1}{(3x+1)^2} \neq \frac{1}{(3x^2+1)}$

Example 3: Let $f(x) = x^2$ *and* $g(x) = \dfrac{1}{3x+1}$. Find the domain of $f \circ g$.

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To be in the domain of $f \circ g$, a number x must be in the domain of g, and the corresponding *g(x)* value must be in the domain of *f.*

Step 1: Find the domain of the "inside" (input) function. If there are any restrictions on the domain, **keep them.**

Step 2: Construct the composite function. Find the domain of this new function. If there are any restrictions on this domain, **add them to the restrictions from Step 1**. If there is an overlap, use the more restrictive domain.

Find the domain of $f \circ g$. $f(g(x))$

1. Find domain of inside function $g(x) = \dfrac{1}{3x+1}$

with fractions, denominator $\neq 0$

$$3x+1 \neq 0$$
$$\underline{\quad -1 \quad -1}$$
$$\frac{3x}{3} \neq \frac{-1}{3}$$
$$x \neq -1/3$$

Domain: $\{x \mid x \neq -1/3\}$

2. Calculate $f(g(x))$

$$f\left(\frac{1}{3x+1}\right) \rightarrow \left(\frac{1}{3x+1}\right)^2 = \frac{1}{(3x+1)^2} = \frac{1}{9x^2+6x+1}$$

Find domain of $\dfrac{1}{9x^2+6x+1}$

$$9x^2+6x+1 \neq 0$$
$$(3x+1)(3x+1) \neq 0$$
$$(3x+1)^2 \neq 0$$

$$\sqrt{(3x+1)^2} \neq \sqrt{0}$$
$$3x+1 \neq 0$$
$$\underline{\quad -1 \quad -1}$$
$$3x \neq -1$$
$$\frac{3x}{3} \neq \frac{-1}{3}$$

$$\boxed{\{x \mid x \neq \tfrac{-1}{3}\}}$$

Example 4: Find $f \circ g$ *and* $g \circ f$ and the domain of each.

$$f(x) = \frac{3x}{x-1} \qquad g(x) = \frac{2}{x}$$

Find $f \circ g$. State the domain.

① $f \circ g = f(g(x))$ Domain of $g(x) = \frac{2}{x}$ $x \neq 0$

$\{x \mid x \neq 0\}$

② Find $f(g(x))$

$f\left(\frac{2}{x}\right)$ $\frac{3\left(\frac{2}{x}\right)}{\left(\frac{2}{x}\right) - 1} = \frac{\frac{6}{x}}{\frac{2}{x} - \frac{x}{x}} = \frac{\frac{6}{x}}{\frac{2-x}{x}} \div$

$\frac{6}{\cancel{x}} \cdot \frac{\cancel{x}}{2-x} = \frac{6}{2-x} = f(g(x))$

$2 - x \neq 0$

$-2 \quad -2$

$\frac{-x}{-1} \neq \frac{-2}{-1}$ $x \neq 2$

Domain of $f(g(x)) = \{x \mid x \neq 0, x \neq 2\}$

Find $g \circ f$. State the domain.

① $g(f(x))$ Domain of $f(x) = \frac{3x}{x-1}$ $x - 1 \neq 0$

$+1 \quad +1$

$x \neq 1$

② Find $g(f(x))$

$g\left(\frac{3x}{x-1}\right) \rightarrow \frac{2}{\frac{3x}{x-1}} \div \frac{2}{1} \cdot \frac{x-1}{3x} = \frac{2(x-1)}{3x}$

$g(f(x)) = \frac{2x-2}{3x}$ $\frac{3x}{3} \neq \frac{0}{3}$ $x \neq 0$

Domain of $g(f(x)) = \{x \mid x \neq 0, x \neq 1\}$

Example 5. Let $S(x,y)=\left(x,\ \frac{y}{3}\right)$ *and* $T(x,y)=(x-1,\ y-2)$

a) Describe S and T in words.

S: vertical shrink by a factor of 1/3

T: left 1
down 2

b) Write a simplified formula for the composite $(T\circ S)(x,\ y)$ and describe it in words.

$(T\circ S)(x,y) \rightarrow T(S(x,y))$

$T\left(x, \frac{y}{3}\right)$

$\left(x-1, \frac{y}{3}-2\right)$

vertical shrink by 1/3
followed by left 1, down 2

c) Write a simplified formula for the composite $(S\circ T)(x,\ y)$ and describe it in words.

$(S\circ T)(x,y) \rightarrow S(T(x,y))$

$S(x-1, y-2)$

$\left(x-1, \frac{y-2}{3}\right)$

Translation left 1, down 2 followed by vertical shrink of 1/3